MATH 218: Elementary Linear Algebra with Applications

Spring 2015-2016, Quiz 1, Duration: 60 min.

Name: Solution

Exercise	Points	Scores
1	18	
2	17	
3	15	
4	20	
5	15	
6	15	
Total	100	

INSTRUCTIONS:

(a) Explain your answers in detail and clearly to ensure full credit.

(b) No book. No notes. No calculator.

Exercise 1. (18 points) Find all the solutions of the following system

$$\begin{cases} x_1 + x_2 + x_4 - x_5 &= 1 \\ -x_1 - x_2 + x_3 - 3x_4 &= 2 \\ 2x_1 + 2x_2 + x_3 - x_5 &= 1 \\ x_1 + x_2 + x_3 - x_4 - x_5 &= 2 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & -3 & 0 & 2 \\ 2 & 2 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[R_4 - R_1 \to R_4]{R_2 + R_1 \to R_2,\ R_3 - 2R_1 \to R_3} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_4 - R_2 \to R_4]{R_3 - R_2 \to R_3} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_4 - R_3 \to R_4} \left(\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & \boxed{1} & -2 & -1 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow$$

x_1, x_3, x_5 leading variables

$x_2 = t,\ x_4 = s$ free variables

$$\Rightarrow \quad x_5 = -2$$

$$x_4 = s$$

$$x_3 = 3 + 2s - 2 = 1 + 2s$$

$$x_2 = t$$

$$x_1 = 1 - t - s - 2 = -1 - t - s$$

$\Rightarrow$ Solutions are $\begin{pmatrix} -1-t-s \\ t \\ 1+2s \\ s \\ -2 \end{pmatrix}, \quad s,t \in \mathbb{R}$

Exercise 2. Consider the following linear system

$$\begin{cases} x_2 - x_3 + x_4 & = -1 \\ x_1 - ax_2 + 2x_3 & = 2 \\ 2x_1 + x_2 + 4x_3 & = 1 \\ 2x_1 + 2x_2 + 3x_3 + x_4 & = b \end{cases}$$

Determine the values of a and b for which the system

(a) **(10 points)** is inconsistent.

$$\left(\begin{array}{cccc|c} 0 & 1 & -1 & 1 & -1 \\ 1 & -a & 2 & 0 & 2 \\ 2 & 1 & 4 & 0 & 1 \\ 2 & 2 & 3 & 1 & b \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cccc|c} 1 & -a & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 2 & 1 & 4 & 0 & 1 \\ 2 & 2 & 3 & 1 & b \end{array}\right)$$

$$\xrightarrow[R_4 - 2R_1 \to R_4]{R_3 - 2R_1 \to R_3} \left(\begin{array}{cccc|c} 1 & -a & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1+2a & 0 & 0 & -3 \\ 0 & 2+2a & -1 & 1 & b-4 \end{array}\right)$$

$$\xrightarrow[R_4 - (2+2a)R_2) \to R_4]{R_3 - (1+2a)R_2 \to R_3} \left(\begin{array}{cccc|c} 1 & -a & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1+2a & -(1+2a) & -2+2a \\ 0 & 0 & 1+2a & -(1+2a) & b-2+2a \end{array}\right)$$

$$\xrightarrow{R_4 - R_3 \to R_4} \left(\begin{array}{cccc|c} 1 & -a & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1+2a & -(1+2a) & -2+2a \\ 0 & 0 & 0 & 0 & b \end{array}\right)$$

So → $b \neq 0$ inconsistent $\left(\begin{array}{cccc|c} & & & & \\ & & & & \\ 0 & 0 & 0 & 0 & \ast \end{array}\right)$ — not zero

So → $b = 0$ → $a \neq -\frac{1}{2}$ $\left(\begin{array}{cccc|c} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ ∞ solutions since one free variable.

$b = 0$ → $a = -\frac{1}{2}$ inconsistent:

$$\left(\begin{array}{cccc|c} & & & & \\ & & & & \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Sol: For $b \neq 0$

and $a = -\frac{1}{2}, b = 0$

(b) **(5 points)** has infinitely many solutions.

For $b = 0$ and $a \neq -\frac{1}{2}$

(c) **(2 points)** has a unique solution.

Never.

Exercise 3. **(15 points)** Consider the following matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ a & a & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Determine the value(s) of a for which the matrix system $A^3X = 0$ has a unique solution.

$A^3X = 0$ has unique solution $\iff A^3$ invertible

$\iff |A^3| \neq 0$

$\iff |A|^3 \neq 0$

$\iff |A| \neq 0$

Compute $|A| = 3a - 1$

So:

$A^3X = 0$ has unique solution $\iff a \neq \frac{1}{3}$

Exercise 4. Prove or disprove (via a counterexample) the following statements[1]:

(a) **(5 points)** If A, B are two square matrices such that A and $A+B$ are symmetric then B is symmetric.

Write $B = A + B - A$

$B^t = (A + B - A)^t$) transpose property

$= (A+B)^t - A^t$) since $A^t = A$ and $(A+B)^t = A+B$

$= A+B - A$

$= B \Rightarrow$ <u>B symmetric.</u>

(b) **(5 points)** Let A be $n \times n$ a matrix. If the system $AX = -X$ (X being a vector), with $A \neq 0$, admits a unique solution then A is invertible.

Notice $AX = X \iff (A+I)X = 0$

So $AX = X$ unique solution $\iff (A+I)X$ has unique solution.

$\iff A+I$ invertible.

$\Rightarrow$ <u>Statement is wrong</u> for matrix

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ invertible

So $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X = -X$ has unique solution but

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ not invertible.

[1] either the statement is true and you have to prove it <u>or</u> the statement is wrong and you have to provide a counterexample

(c) **(5 points)** If A is a 5×5 matrix such that $A = -A^t$ then A is not invertible.

True: Indeed let A 5×5 s.t. $A = -A^t$

so $|A| = |-A^t| \underset{\text{determinant property}}{=} (-1)^5 |A^t| \underset{\text{determinant of transpose.}}{=} -|A|$

$\Rightarrow |A| = 0$

$\Rightarrow$ A is not invertible.

(d) **(5 points)** Let A, B be two square matrices. Then $(A - B)(A + B) = A^2 - B^2$.

Note that $(A-B)(A+B) = A^2 + \underbrace{AB - BA}_{\text{not } = 0 \text{ in general}} - B^2$

Indeed take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ So for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

we have $(A-B)(A+B) \neq A^2 - B^2$

So statement is wrong

Exercise 5. A $n \times n$ matrix A is called *idempotent* if $A^2 = A$.

(1) **(2 points)** Is I idempotent? Justify your answer.

$I^2 = I \cdot I = I.$

(2) **(3 points)** Show that I is the only invertible idempotent matrix.

Let A invertible and $A^2 = A$

$\Rightarrow A^2 A^{-1} = A A^{-1}$

$\Rightarrow A A A^{-1} = I$

$\Rightarrow A = I$

So I is the only invertible idempotent matrix

(3) **(10 points)** Let A be a $n \times n$ idempotent matrix and let P be a $n \times n$ invertible matrix. Show that PAP^{-1} is idempotent.

Compute $PA\cancel{P^{-1}}(\cancel{P}AP^{-1})$ since $P^{-1}P = I$

$= PAIAP^{-1}$ since $AI = A$

$= PAAP^{-1}$

$= PA^2P^{-1}$ since $A^2 = A.$

$= PAP^{-1}$

So $PAP^{-1}PAP^{-1} = PAP^{-1}$

So PAP^{-1} is idempotent.

Exercise 6. (15 points) Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and assume its determinant is 2. Compute the determinant of $\begin{pmatrix} -d & e & f \\ -g & h & i \\ -3a+2g & 3b-2h & 3c-2i \end{pmatrix}$.

$$\begin{vmatrix} -d & e & f \\ -g & h & i \\ -3a+2g & 3b-2h & 3c-2i \end{vmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} - \begin{vmatrix} -3a+2g & 3b-2h & 3c-2i \\ -g & h & i \\ -d & e & f \end{vmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} + \begin{vmatrix} -3a+2g & 3b-2h & 3c-2i \\ -d & e & f \\ -g & h & i \end{vmatrix}$$

$$\xrightarrow{-C_1 \rightarrow C_1} - \begin{vmatrix} 3a-2g & 3b-2h & 3c-2i \\ d & e & f \\ g & h & i \end{vmatrix}$$

$R_1 + 2R_3 \to R_1$

$$\xrightarrow{\quad} - \begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$\frac{1}{3}R_1 \to R_1$

$$\xrightarrow{\quad} -3 \underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{=2} = \underline{-6}$$